

MEASURABLE BUNDLES OF BANACH ALGEBRAS

INOMJON GANIEV, FARRUKH MUKHAMEDOV,
AND KARIMBERGEN KUDAYBERGENOV

ABSTRACT. In the present paper we investigate Banach–Kantorovich algebras over faithful solid subalgebras of algebras measurable functions. We prove that any Banach–Kantorovich algebra over faithful solid subalgebras of algebra measurable functions represented as a measurable bundle of Banach algebras with vector-valued lifting. We apply such representation to the spectrum of elements Banach–Kantorovich algebras.

Keywords: measurable bundle; Banach–Kantorovich module; lattice; Banach–Mazur Theorem.

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1. INTRODUCTION

It is known that the theory of Banach bundles stemming from paper [17], where it is was shown that such a theory had vast applications in analysis. The study of Banach lattices in terms of sections of continuous Banach bundles has been started by Giertz (see [12]). Later Gutman [13] created the theory of continuous Banach bundles and measurable Banach bundles admitting lifting [14]. A portion of the Gutman’s theory was specified in the case of bundles of measurable Banach lattices by Ganiev [7] and Kusraev [16].

Nowadays the methods of Banach bundles has many applications in the operator algebras [1, 2, 6]. In [8] it was considered C^* -algebras over ring of all measurable functions and it has been shown that any C^* -algebra over a ring measurable functions can be represented as a measurable bundle of C^* -algebras. Some application of this representation to ergodic theorems have been studied in [11].

It is known [19] that one of the important results in the theory of C^* -algebras is the Gelfand–Naimark’s theorem, which describes commutative C^* -algebras over the complex field \mathbb{C} as an algebra of complex valued continuous functions defined on the set of all pure states of given C^* -algebra. In [3] it has been proved a vector version of the Gelfand–Naimark’s theorem for commutative C^* -algebras over a ring measurable functions. GNS-representation for such C^* -algebras was obtained in [4].

In section 2 we consider a Banach–Kantorovich algebra over a faithful solid subalgebras of the algebra measurable functions. We prove a Banach–Kantorovich algebra over a faithful solid subalgebras represented as measurable bundle of banach algebras. Note that in [10] C^* -algebras over ideals of L_0 have been considered.

In section 3 we prove a vector version of the Gelfand–Mazur’s theorem.

2. MEASURABLE BUNDLES OF BANACH ALGEBRAS

Let (Ω, Σ, μ) be a measure space with a finite measure μ and let $L^0(\Omega)$ be the algebra of equivalence classes of all complex measurable functions on Ω . Let $L^\infty(\Omega)$ be the algebra of all equivalence classes of bounded complex measurable functions on Ω with the norm $\|\hat{f}\|_{L^\infty(\Omega)} = \inf\{\alpha > 0 : |\hat{f}| \leq \alpha \mathbf{1}\}$, here $\mathbf{1}$ is the unit function, i.e. $\mathbf{1}(\omega) = 1$ for all $\omega \in \Omega$.

A complex linear space X is said to be normed by $L^0(\Omega)$ if there is a map $\|\cdot\| : X \rightarrow L^0(\Omega)$ such that for any $x, y \in X$, $\lambda \in \mathbb{C}$ the following conditions are fulfilled:

- (1) $\|x\| \geq 0$; $\|x\| = 0 \Leftrightarrow x = 0$;
- (1) $\|\lambda x\| = |\lambda| \|x\|$;
- (3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(X, \|\cdot\|)$ is called a *lattice-normed* space over $L^0(\Omega)$. A lattice-normed space X is called *d-decomposable* if for any $x \in X$ with $\|x\| = \lambda_1 + \lambda_2$, $0 \leq \lambda_1, \lambda_2 \in L^0(\Omega)$, $\lambda_1 \lambda_2 = 0$ there exist $x_1, x_2 \in X$ such that $x = x_1 + x_2$ and $\|x_k\| = \lambda_k$, $k = 1, 2$.

A net (x_α) in X is *(bo)-converging* to $x \in X$, if $\|x - x_\alpha\| \xrightarrow{(o)} 0$ (note that the order convergence in $L^0(\Omega)$ coincides with convergence almost everywhere). A lattice-normed space X which is *d-decomposable* and complete with respect to *(bo)-convergence* is called a *Banach-Kantorovich space*. It is known that every Banach-Kantorovich space X over $L^0(\Omega)$ is a module over $L^0(\Omega)$ and $\|ax\| = |a| \|x\|$ for all $a \in L^0(\Omega)$, $x \in X$ (see [15]).

Let E be a *faithful solid subalgebra* in $L^0(\Omega)$, i.e. the inequality $|x| \leq |y|$ implies $x \in E$ for arbitrary $x \in L^0(\Omega)$, $y \in E$ and $E^\perp = \{0\}$. Note that one has $L^\infty \subset E \subset L^0(\Omega)$. Consider an arbitrary algebra \mathcal{U} over the field \mathbb{C} such that \mathcal{U} is a module over E , i.e. $(au)v = a(uv) = u(av)$ for all $a \in E$, $u, v \in \mathcal{U}$. Consider E -valued norm $\|\cdot\|$ on \mathcal{U} which endows \mathcal{U} with Banach-Kantorovich structure, in particular, one has $\|au\| = |a| \|u\|$ for all $a \in E$, $u \in \mathcal{U}$.

An algebra \mathcal{U} is called *Banach-Kantorovich algebra* over E , if for every $u, v \in \mathcal{U}$ one has $\|uv\| \leq \|u\| \|v\|$. If \mathcal{U} is a Banach-Kantorovich algebra over E with unit e such that $\|e\| = \mathbf{1}$, where $\mathbf{1}$ is the unit in E , then \mathcal{U} is called *unital Banach-Kantorovich algebra*.

EXAMPLE. Let us provide an example of Banach-Kantorovich algebra over E . To do this, let us recall some definitions taken from [4]. Consider a modulus \mathcal{A} over E , here as before, E stands for faithful solid subalgebra of $L^0(\Omega)$. A mapping $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \rightarrow E$ is called *E-valued inner product*, if for every $x, y, z \in \mathcal{A}$, $l \in E$ one has $\langle x, x \rangle \geq 0$; $\langle x, x \rangle = 0$ if and only if $x = 0$; $\langle x, y \rangle = \overline{\langle y, x \rangle}$; $\langle lx, y \rangle = l \langle x, y \rangle$; $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

If $\langle \cdot, \cdot \rangle : \mathcal{A} \times \mathcal{A} \rightarrow E$ is a *E-valued inner product*, the formula $\|x\| = \sqrt{\langle x, x \rangle}$ defines a *d-decomposable E-valued norm* on \mathcal{A} . Then the pair $(\mathcal{A}, \langle \cdot, \cdot \rangle)$ is called *Hilbert-Kaplansky modules*, if $(\mathcal{A}, \|\cdot\|)$ is BKS over E . Let A and F be BKS over E . An operator $T : A \rightarrow F$ is called *E-linear*, if one has $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for every $x, y \in A$, $\alpha, \beta \in E$. A linear operator T is called *E-bounded* if there exists $c \in E$ such that $\|T(x)\| \leq c \|x\|$ for every $x \in A$. For *E-linear* and *E-bounded* operator T one defines $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$, which is a norm of T (see [15]).

Now let \mathcal{A} be a Hilbert-Kaplansky modulus over E . By $B(\mathcal{A})$ we denote the set of E -linear, E -bounded operators on the Hilbert-Kaplansky modules \mathcal{A} over E . Then $B(\mathcal{A})$ is a Banach-Kantorovich algebra over E .

We shall consider a map $\mathcal{X} : \Omega \rightarrow X(\omega)$, where $X(\omega) \neq \{0\}$, is a Banach algebra for all $\omega \in \Omega$. A function u is called a *section* of \mathcal{X} if it is defined on Ω almost everywhere and takes a value $u(\omega) \in X(\omega)$ for all $\omega \in \text{dom } u$, where $\text{dom } u$ is the domain of u . Let L be some set of sections.

A pair (\mathcal{X}, L) is called a *measurable bundle of banach algebras*, if

- (1) $\lambda_1 c_1 + \lambda_2 c_2 \in L$ for all $\lambda_1, \lambda_2 \in \mathbb{C}$, $c_1, c_2 \in L$, where $\lambda_1 c_1 + \lambda_2 c_2 : \omega \in \text{dom } c_1 \cap \text{dom } c_2 \rightarrow \lambda_1 c_1(\omega) + \lambda_2 c_2(\omega)$;
- (2) the function $c : \omega \in \text{dom } c \rightarrow \|c\|_{X(\omega)}$ is measurable for all $c \in L$;
- (3) the set $\{c(\omega) : c \in L, \omega \in \text{dom } c\}$ is dense in $X(\omega)$ for all $c \in L$;
- (4) $uv \in L$ for all $u, v \in L$, where $uv : \omega \in \text{dom } (u) \cap \text{dom } (v) \rightarrow u(\omega)v(\omega)$.

A section s is called *simple*, if there exists $c_i, A_i \in \Sigma$, $i \in \overline{1, n}$ such that

$$s(\omega) = \sum_{i=1}^n \chi_{A_i} c_i.$$

A section u is called *measurable* if there exists a sequence $(s_n)_{n \in \mathbb{N}}$ of simple sections such that $\|s_n(\omega) - u(\omega)\|_{X(\omega)} \rightarrow 0$ for almost all $\omega \in \Omega$. We denote by $M(\Omega, \mathcal{X})$ the set of all measurable sections and $L^0(\Omega, \mathcal{X})$ denotes the factorization of this set with respect to equality almost everywhere. By \hat{u} we denote the class from $L^0(\Omega, \mathcal{X})$, containing section $u \in M(\Omega, \mathcal{X})$. A function $\omega \rightarrow \|u(\omega)\|_{X(\omega)}$ is measurable for all $u \in M(\Omega, \mathcal{X})$. By $\|\hat{u}\|$ we denote the element in $L^0(\Omega)$, containing the function $\|u(\omega)\|_{X(\omega)}$. For $\hat{u}, \hat{v} \in L^0(\Omega)$ we put $\hat{u} \cdot \hat{v} = \widehat{u(\omega) \cdot v(\omega)}$.

Set

$$E(\Omega, \mathcal{X}) = \{x \in L^0(\Omega, \mathcal{X}) : \|x\| \in E\}.$$

It is known [14] that $E(\Omega, \mathcal{X})$ is a Banach-Kantorovich space over E . Since $X(\omega)$ is a Banach algebra we get

$$\begin{aligned} \|\hat{u}\hat{v}\|(\omega) &= \|u(\omega)v(\omega)\|_{X(\omega)} \leq \|u(\omega)\|_{X(\omega)}\|v(\omega)\|_{X(\omega)} \\ &= \|u(\omega)\|_{X(\omega)}\|v(\omega)\|_{X(\omega)} = \|\hat{u}\|\|\hat{v}\|(\omega) \end{aligned}$$

for almost all $\omega \in \Omega$. Thus $\|\hat{u}\hat{v}\| \leq \|\hat{u}\|\|\hat{v}\|$. Hence, $(E(\Omega, \mathcal{X}), \|\cdot\|)$ is a Banach-Kantorovich algebra over E .

So, we obtain the following

Proposition 2.1. *If \mathcal{X} is a measurable bundle of Banach algebras, then $(E(\Omega, \mathcal{X}), \|\cdot\|)$ is a Banach-Kantorovich algebra over E .*

Let $\mathcal{L}^\infty(\Omega)$ be the set of all bounded measurable functions on Ω with the norm

$$\|f\|_{\mathcal{L}^\infty(\Omega)} = \inf\{\alpha > 0 : |f(\omega)| \leq \alpha \text{ for almost all } \omega \in \Omega\}.$$

As before, by $L^\infty(\Omega)$ stands for the algebra of all equivalence classes of bounded complex measurable functions on Ω with the norm

$$\|\hat{f}\|_{L^\infty(\Omega)} = \inf\{\alpha > 0 : |\hat{f}| \leq \alpha \mathbf{1}\}.$$

Set

$$\mathcal{L}^\infty(\Omega, \mathcal{X}) = \{u \in M(\Omega, \mathcal{X}) : \|u(\omega)\|_{X(\omega)} \in \mathcal{L}^\infty(\Omega)\}$$

and

$$L^\infty(\Omega, \mathcal{X}) = \{\hat{u} \in L^0(\Omega, \mathcal{X}) : \|\hat{u}\| \in L^\infty(\Omega)\}.$$

One can define the spaces $\mathcal{L}^\infty(\Omega, \mathcal{X})$ and $L^\infty(\Omega, \mathcal{X})$ with real-valued norms $\|u\|_{\mathcal{L}^\infty(\Omega, \mathcal{X})} = \sup_{\omega \in \Omega} \|u(\omega)\|_{X(\omega)}$ and $\|\hat{u}\|_\infty = \left\| \|\hat{u}\| \right\|_{L^\infty(\Omega)}$, respectively.

It is known [14], [15] that there is a homomorphism $p : L^\infty(\Omega) \rightarrow \mathcal{L}^\infty(\Omega)$ being a lifting such that

1. $p(\hat{f}) \in \hat{f}$ and $\text{dom } p(\hat{f}) = \Omega$;
2. $\|p(\hat{f})\|_{\mathcal{L}^\infty(\Omega)} = \|\hat{f}\|_{L^\infty(\Omega)}$.

The homomorphism p is usually called a *lifting* from $L^\infty(\Omega)$ to $\mathcal{L}^\infty(\Omega)$.

The map $\ell_{\mathcal{X}} : L^\infty(\Omega, \mathcal{X}) \rightarrow \mathcal{L}^\infty(\Omega, \mathcal{X})$ is called a *vector-valued lifting* (associated with p), if for all $\hat{u}, \hat{v} \in L^\infty(\Omega, \mathcal{X})$ and $a \in L^\infty(\Omega)$ the following conditions are satisfied:

1. $\ell_{\mathcal{X}}(\hat{u}) \in \hat{u}$, $\text{dom } (\ell_{\mathcal{X}}(\hat{u})) = \Omega$;
2. $\|\ell_{\mathcal{X}}(\hat{u})(\omega)\|_{X(\omega)} = p(\|\hat{u}\|)(\omega)$;
3. $\ell_{\mathcal{X}}(\hat{u} + \hat{v}) = \ell_{\mathcal{X}}(\hat{u}) + \ell_{\mathcal{X}}(\hat{v})$;
4. $\ell_{\mathcal{X}}(a\hat{u}) = p(a)\ell_{\mathcal{X}}(\hat{u})$;
5. $\ell_{\mathcal{X}}(\hat{u}\hat{v}) = \ell_{\mathcal{X}}(\hat{u})\ell_{\mathcal{X}}(\hat{v})$;
6. for every $\omega \in \Omega$ the set $\{\ell_{\mathcal{X}}(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, \mathcal{X})\}$ is dense in $X(\omega)$.

Let \mathcal{X} and \mathcal{Y} be measurable bundles of banach algebras over Ω . Assume that for each $\omega \in \Omega$ the mapping $H_\omega : X(\omega) \rightarrow Y(\omega)$ is an injective homomorphism of banach algebras. A mapping $H : \omega \rightarrow H_\omega$ is called *inclusion* of \mathcal{X} into \mathcal{Y} if one has

$$\{H_\omega(u(\omega)) : u \in M(\Omega, \mathcal{X})\} \subset M(\Omega, \mathcal{Y}).$$

If $\{H_\omega(u(\omega)) : u \in M(\Omega, \mathcal{X})\} = M(\Omega, \mathcal{Y})$ the inclusion H is called *isomorphism* from \mathcal{X} onto \mathcal{Y} . In this case, the bundles \mathcal{X} and \mathcal{Y} are called *isomorphic*.

Theorem 2.2. *For every Banach–Kantorovich algebra \mathcal{U} over E there exists a unique (up to isomorphism) measurable bundle of banach algebras (\mathcal{X}, L) with a vector-valued lifting $\ell_{\mathcal{X}}$ such that \mathcal{U} is isometrically isomorphic to $E(\Omega, \mathcal{X})$, and one has*

$$\{\ell_{\mathcal{X}}(x)(\omega) : x \in L^\infty(\Omega, \mathcal{X})\} = X(\omega)$$

for all $\omega \in \Omega$. Moreover, if \mathcal{U} is a unital algebra, then $X(\omega)$ is also a unital algebra for all $\omega \in \Omega$.

Proof. Put

$$\mathcal{U}_b = \{u \in \mathcal{U} : \|u\| \in L^\infty(\Omega)\}.$$

It is clear that \mathcal{U}_b is an $L^\infty(\Omega)$ -module and (bo)-complete in \mathcal{U} .

On the other hand, \mathcal{U}_b is a Banach algebra with respect to the norm

$$\|u\|_\infty = \|\|u\|\|_{L^\infty(\Omega)}, \quad u \in \mathcal{U}_b.$$

Define a seminorm α_ω on \mathcal{U}_b by $\alpha_\omega(u) = p(\|u\|)(\omega)$ for all $\omega \in \Omega$, where p is the lifting on $L^\infty(\Omega)$. Set

$$I_\omega = \{u \in \mathcal{U}_b : \alpha_\omega(u) = 0\}.$$

Let us consider a sequence $(u_n)_{n \in \mathbb{N}}$ in I_ω such that $\|u_n - u\|_\infty \rightarrow 0$ for some $u \in \mathcal{U}_b$. Then

$$\alpha_\omega(u) \leq \alpha_\omega(u_n - u) + \alpha_\omega(u_n) \leq \|u_n - u\|_\infty \rightarrow 0.$$

Therefore, $u \in I_\omega$. It is clear that $\lambda_1 u + \lambda_2 v \in I_\omega$ for all $u, v \in I_\omega, \lambda_1, \lambda_2 \in \mathbb{C}$. For $u \in I_\omega, v \in \mathcal{U}_b$ one gets

$$\alpha_\omega(u \cdot v) = p(\|u \cdot v\|)(\omega) \leq p(\|u\|)(\omega)p(\|v\|)(\omega) = \alpha_\omega(u)p(\|v\|)(\omega) = 0,$$

i.e. $u \cdot v \in I_\omega$. Hence, I_ω is a closed ideal in $(\mathcal{U}_b, \|\cdot\|_\infty)$.

Let $X(\omega) = \mathcal{U}_b/I_\omega$ be a factor-algebra and $i_\omega : \mathcal{U}_b \rightarrow X(\omega)$ be the natural homomorphism from \mathcal{U}_b onto $X(\omega)$. Then $X(\omega)$ is a banach algebra with respect to the norm

$$\|i_\omega(u)\|_\omega^0 = \inf\{\|v\|_\infty : u - v \in I_\omega\}, u \in \mathcal{U}_b, \omega \in \Omega.$$

Let $\|\cdot\|_\omega$ be the norm on $X(\omega)$ generated by the seminorm α_ω , i.e. $\|i_\omega(u)\|_\omega = \alpha_\omega(u)$, $u \in \mathcal{U}_b$ for all $\omega \in \Omega$.

Let us show that

$$\|i_\omega(u)\|_\omega = \|i_\omega(u)\|_\omega^0, u \in \mathcal{U}_b, \omega \in \Omega.$$

Indeed, fix $\omega \in \Omega$ and $u \in \mathcal{U}_b$. If $v \in \mathcal{U}_b$ and $u - v \in I_\omega$, then

$$\|i_\omega(u)\|_\omega = \alpha_\omega(u) \leq \alpha_\omega(v) + \alpha_\omega(u - v) = \alpha_\omega(v) \leq \|v\|_\infty.$$

Whence $\|i_\omega(u)\|_\omega \leq \|i_\omega(u)\|_\omega^0$.

To show the converse inequality we take an arbitrary $\varepsilon > 0$. Set

$$A_\varepsilon = \{\omega' \in \Omega : p(\|u\|)(\omega') \leq \alpha_\omega(u) + \varepsilon\}$$

and

$$\pi_\varepsilon = \chi_{A_\varepsilon}, \quad u_\varepsilon = \pi_\varepsilon u.$$

Then

$$(2.1) \quad \pi_\varepsilon \|u\| \leq (\alpha_\omega(u) + \varepsilon) \pi_\varepsilon$$

and

$$\pi_\varepsilon^\perp \|u\| \geq (\alpha_\omega(u) + \varepsilon) \pi_\varepsilon^\perp.$$

The last inequality implies that

$$p(\pi_\varepsilon^\perp \|u\|)(\omega) \geq (\alpha_\omega(u) + \varepsilon) p(\pi_\varepsilon^\perp)(\omega),$$

i.e.

$$p(\pi_\varepsilon^\perp)(\omega)(p(\|u\|)(\omega) - \alpha_\omega(u)) \geq \varepsilon p(\pi_\varepsilon^\perp)(\omega).$$

Hence, $0 \geq \varepsilon p(\pi_\varepsilon^\perp)(\omega)$ or $p(\pi_\varepsilon^\perp)(\omega) = 1$. Therefore,

$$\begin{aligned} \alpha_\omega(u - u_\varepsilon) &= \alpha_\omega(\pi_\varepsilon^\perp u) = p(\|\pi_\varepsilon^\perp u\|) = \\ &= p(\pi_\varepsilon^\perp)(\omega) p(\|u\|)(\omega) = \alpha_\omega(u) = 0. \end{aligned}$$

Consequently, $u - u_\varepsilon \in I_\omega$. It follows from (2.1) that

$$\|u_\varepsilon\| = \|\pi_\varepsilon u\| = \pi_\varepsilon \|u\| \leq (\alpha_\omega(u) + \varepsilon) \pi_\varepsilon.$$

This means that $\|u_\varepsilon\|_\infty \leq \alpha_\omega(u) + \varepsilon$. Since $\varepsilon > 0$ be an arbitrary we get

$$\|i_\omega(u)\|_\omega^0 \leq \|i_\omega(u)\|_\omega + \varepsilon,$$

i.e. $\|i_\omega(u)\|_\omega^0 \leq \|i_\omega(u)\|_\omega$.

Now let us define a mapping \mathcal{X} which assigns for each $\omega \in \Omega$ the banach algebra $X(\omega)$. By L we denote the set of all sections of the form $\omega \in \Omega : \omega \rightarrow i_\omega(u)$, where $u \in \mathcal{U}_b$. One can see that (\mathcal{X}, L) is a measurable bundle of banach algebras.

Let us consider $E(\Omega, \mathcal{X})$ with E -valued norm $\|\cdot\|_{E(\Omega, \mathcal{X})}$. Let us show that \mathcal{U} is isometrically isomorphic to $E(\Omega, \mathcal{X})$.

For each $u \in \mathcal{U}_b$ define $\tau(u) = \widehat{i_\omega(u)}$. Then for $u \in \mathcal{U}_b$ and $\omega \in \Omega$ one has

$$\|i_\omega(u)\|_\omega = \alpha_\omega(u) = p(\|u\|)(\omega).$$

Hence, $\|\widehat{i_\omega(u)}\|_{E(\Omega, \mathcal{X})} = \|u\|$, and therefore, τ is an isometry from \mathcal{U}_b into $L^\infty(\Omega, \mathcal{X})$. Since $\tau(\mathcal{U}_b)$ contains the set of all simple sections then τ is isometry from \mathcal{U}_b onto $L^\infty(\Omega, \mathcal{X})$. Moreover, we have

$$\tau(u \cdot v) = \widehat{i_\omega(u \cdot v)} = \widehat{i_\omega(u) i_\omega(v)} = \widehat{i_\omega(u)} \widehat{i_\omega(v)} = \tau(u) \tau(v).$$

So, τ is an isometrically isomorphism from \mathcal{U}_b onto $L^\infty(\Omega, \mathcal{X})$. Since \mathcal{U}_b is (bo) -complete in \mathcal{U} we obtain that τ can be extended up to isometrically isomorphism from \mathcal{U} onto $E(\Omega, \mathcal{X})$. Besides, it is clear that τ preserves the multiplication, i.e. τ is an isomorphism of algebras \mathcal{U} and $E(\Omega, \mathcal{X})$.

Now let us establish that (\mathcal{X}, L) is a measurable bundle with a vector-valued lifting. Define a mapping $\ell_{\mathcal{X}} : L^\infty(\Omega, \mathcal{X}) \rightarrow \mathcal{L}^\infty(\Omega, \mathcal{X})$ by

$$\ell_{\mathcal{X}}(\hat{u})(\omega) = i_\omega(\tau^{-1}(\hat{u})), \hat{u} \in L^\infty(\Omega, \mathcal{X}).$$

Since $i_\omega(\tau^{-1}(\hat{u}))$ is defined for all $\omega \in \Omega$, then $\text{dom}(\ell_{\mathcal{X}}(\hat{u})) = \Omega$. For $\hat{u} \in L^\infty(\Omega, \mathcal{X})$ and $\omega \in \Omega$ one has

$$p(\|\hat{u}\|)(\omega) = p(\|\tau^{-1}(\hat{u})\|)(\omega) = \alpha_\omega(\tau^{-1}(\hat{u})) = \|i_\omega(\tau^{-1}(\hat{u}))\|_\omega = \|\ell_{\mathcal{X}}(\hat{u})(\omega)\|_\omega.$$

The linearity of $\ell_{\mathcal{X}}$ is evident. For $\hat{u}, \hat{v} \in L^\infty(\Omega, \mathcal{X})$ we obtain

$$\ell_{\mathcal{X}}(\hat{u} \cdot \hat{v})(\omega) = i_\omega(\tau^{-1}(\hat{u} \cdot \hat{v})) = i_\omega(\tau^{-1}(\hat{u})) i_\omega(\tau^{-1}(\hat{v})) = \ell_{\mathcal{X}}(\hat{u})(\omega) \cdot \ell_{\mathcal{X}}(\hat{v})(\omega).$$

According to the construction one gets $\{\ell_{\mathcal{X}}(\hat{u})(\omega) : \hat{u} \in L^\infty(\Omega, \mathcal{X})\} = X(\omega)$.

Now let us prove the uniqueness of \mathcal{X} . Assume that \mathcal{Y} is a measurable bundle of Banach algebras with a vector-valued lifting $\ell_{\mathcal{Y}}$ such that $E(\Omega, \mathcal{Y})$ is isometrically isomorphic to \mathcal{U} .

Let i be an isometrically isomorphism between $L^\infty(\Omega, \mathcal{X})$ and $L^\infty(\Omega, \mathcal{Y})$. Define a linear operator

$$H_\omega : X(\omega) \rightarrow Y(\omega) \quad (\omega \in \Omega)$$

by

$$H_\omega(\ell_{\mathcal{X}}(\hat{u})(\omega)) = \ell_{\mathcal{Y}}(i(\hat{u}))(\omega), \hat{u} \in L^\infty(\Omega, \mathcal{X}).$$

Then for $\hat{u} \in L^\infty(\Omega, \mathcal{X})$ we have

$$\begin{aligned} \|H_\omega(\ell_{\mathcal{X}}(\hat{u})(\omega))\|_{Y(\omega)} &= \|\ell_{\mathcal{Y}}(i(\hat{u}))(\omega)\|_{Y(\omega)} = p(\|i(\hat{u})\|)(\omega) \\ &= p(\|\hat{u}\|)(\omega) = \|\ell_{\mathcal{X}}(\hat{u})(\omega)\|_{X(\omega)}, \end{aligned}$$

i.e. H_ω is an isometry. By the same argument with properties of vector-valued lifting one yields that H_ω is a homomorphism and $\{H_\omega(u(\omega)) : u \in M(\Omega, \mathcal{X})\} = M(\Omega, \mathcal{Y})$. Hence, \mathcal{X} and \mathcal{Y} are isometrically isomorphic.

Now assume that e is a unit in \mathcal{U} , then $e \in \mathcal{U}_b$. Since $i_\omega : \mathcal{U}_b \rightarrow X(\omega)$ is a homomorphism, then $e_\omega = i_\omega(e)$ is a unit in $X(\omega)$ for all $\omega \in \Omega$. The proof is complete. \square

An operator $\Phi : \mathcal{U} \rightarrow \mathcal{U}$ is called *mixing preserving* if one has

$$\Phi \left(\sum_{n=1}^{\infty} \pi_n x_n \right) = \sum_{n=1}^{\infty} \pi_n \Phi(x_n)$$

for any sequence (x_n) in \mathcal{U} and partition of unity (π_n) in ∇ .

As usual, by $\text{Inv}(\mathcal{U})$ we denote the set of all invertible elements of the algebra \mathcal{U} . For $a, b \in E$ $a \gg b$ means that $a(\omega) > b(\omega)$ for almost all $\omega \in \Omega$.

Proposition 2.3. *Let \mathcal{U} be a unital Banach–Kantorovich algebra over E . Then the following statements hold true:*

(i) *if $x \in \mathcal{U}$, $\|x\| \ll \mathbf{1}$ then the element $e - x$ is invertible and*

$$\|(e - x)^{-1} - e\| \leq \|x\|(\mathbf{1} - \|x\|)^{-1};$$

(ii) *if $x \in \text{Inv}(\mathcal{U})$, $h \in \mathcal{U}$ and $2\|h\| \ll \|x^{-1}\|^{-1}$ then $x + h \in \text{Inv}(\mathcal{U})$ and*

$$(2.2) \quad \|(x + h)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|h\|;$$

(iii) *the mapping $x \in \text{Inv}(\mathcal{U}) \rightarrow x^{-1}$ is continuous and mixing preserving.*

Proof. (i) By the inequality

$$\sum_{n=0}^{\infty} \|x^n\| \leq \sum_{n=0}^{\infty} \|x\|^n = (\mathbf{1} - \|x\|)^{-1}$$

it follows that the series $\sum_{n=0}^{\infty} x^n$ (bo)-converges to some $y \in \mathcal{U}$. The sequence

$$(e - x) \sum_{n=0}^k x^n = e - x^{k+1}$$

simultaneously (bo)-converges to $(e - x)y = y(e - x)$ and e , therefore, the element y is inverse of $e - x$. Furthermore, we have

$$\|(e - x)^{-1} - e\| = \left\| \sum_{n=1}^{\infty} x^n \right\| \leq \|x\|(\mathbf{1} - \|x\|)^{-1}.$$

(ii) Taking into account that $x + h = x(e + x^{-1}h)$, $\|x^{-1}h\| \leq \|x^{-1}\|\|h\| \ll \mathbf{1}$ and property (i) one finds $x + h \in \text{Inv}(\mathcal{U})$ and $\|(x + h)^{-1} - x^{-1}\| \leq 2\|x^{-1}\|^2\|h\|$.

(iii) Inequality (2.2) implies that the mapping

$$x \in \text{Inv}(\mathcal{U}) \rightarrow x^{-1} \in \text{Inv}(\mathcal{U})$$

is continuous.

Let $(x_n)_{n \in \mathbb{N}} \subset \text{Inv}(\mathcal{U})$ and let $(\pi_n)_{n \in \mathbb{N}}$ be a partition of the unity in ∇ . Set

$$x = \sum_{n=1}^{\infty} \pi_n x_n.$$

It is clear that

$$x \left(\sum_{n=1}^{\infty} \pi_n x_n^{-1} \right) = \left(\sum_{n=1}^{\infty} \pi_n x_n^{-1} \right) x = e.$$

Therefore $x^{-1} = \sum_{n=1}^{\infty} \pi_n x_n^{-1}$. This yields that the mapping $x \in \text{Inv}(\mathcal{U}) \rightarrow x^{-1}$ is mixing preserving. \square

For every $x \in \mathcal{U}$ by $\text{sp}(x)$ we denote the set of all $a \in E$ for which the element $ae - x$ is not invertible.

Proposition 2.4. *For every $x \in \mathcal{U} \equiv E(\Omega, \mathcal{X})$ the set $\text{sp}(x)$ is non-empty.*

Proof. Without loss of generality we can assume that $\|x\| \leq 1$, because

$$\text{sp}(x) = \{(\mathbf{1} + \|x\|)a : a \in \text{sp}((\mathbf{1} + \|x\|)^{-1}x)\}.$$

Then

$$\|\ell_{\mathcal{X}}(x)(\omega)\|_{X(\omega)} \leq 1$$

for all $\omega \in \Omega$. Hence, for each $\omega \in \Omega$ there exists $\lambda_{\omega} \in \mathbb{C}, |\lambda_{\omega}| \leq 1$ such that

$$(2.3) \quad \lambda_{\omega} \in \text{sp}(\ell_{\mathcal{X}}(x)(\omega)).$$

Now suppose that $\text{sp}(x) = \emptyset$. Denote

$$\mathbf{D} = \{a \in E : |a| \leq 1\}.$$

Since the mapping $a \in \mathbf{D} \rightarrow (ae - x)^{-1}$ is continuous and mixing preserving, then there exists

$$\sup\{\|(ae - x)^{-1}\| : a \in \mathbf{D}\} = c \in E.$$

Now take a nonzero $\pi \in \nabla$ with $\pi c \in L^{\infty}(\Omega)$. Then the set

$$\Omega_0 = \{\omega \in \Omega : p(\pi)(\omega) = 1\}$$

has a positive measure. Fix $\omega \in \Omega_0$. By definition we have

$$\pi\|(ae - x)^{-1}\| \leq \pi c$$

for all $a \in \mathbf{D}$. Therefore, $\pi(ae - x)^{-1} \in L^{\infty}(\Omega, \mathcal{X})$ for every $a \in \mathbf{D}$. Now applying the lifting $\ell_{\mathcal{X}}$ to the equality $\pi(ae - x)^{-1}(ae - x) = \pi e$ one finds

$$\ell_{\mathcal{X}}(\pi(ae - x)^{-1})(\omega)(p(a)(\omega)\ell_{\mathcal{X}}(e)(\omega) - \ell_{\mathcal{X}}(x)(\omega)) = \ell_{\mathcal{X}}(e)(\omega).$$

This implies that the element $p(a)(\omega)e_{\omega} - \ell_{\mathcal{X}}(x)(\omega)$ is invertible in $X(\omega)$ for all $a \in \mathbf{D}$. Due to properties of lifting p we obtain

$$\{p(a)(\omega) : a \in \mathbf{D}\} = \{\lambda_{\omega} \in \mathbb{C} : |\lambda_{\omega}| \leq 1\}.$$

So, every $\lambda_{\omega} \in \mathbb{C}$ with $|\lambda_{\omega}| \leq 1$ does not belong to $\text{sp}(\ell_{\mathcal{X}}(x)(\omega))$ for all $\omega \in \Omega_0$. This contradicts to (2.3), which yields the desired assertion. The proof is complete. \square

By ∇ we denote the Boolean algebra of all idempotents of E . Let \mathcal{U} be a unital Banach–Kantorovich algebra over E , then the subalgebra

$$\pi\mathcal{U} = \{\pi x : x \in \mathcal{U}\}, \pi \in \nabla, \pi \neq 0$$

is considered as unital with unit πe .

By $\text{spm}(x)$ we denote the set of all $a \in E$ such that for each $\pi \in \nabla, \pi \neq 0$ one has $\pi(ae - x) \notin \text{Inv}(\pi\mathcal{U})$. It is clear that $\text{sp}(x) \subset \text{spm}(x)$.

Next result is a variant of the theorem about spectrum for elements of Banach–Kantorovich algebra over E .

Theorem 2.5. *For every $x \in \mathcal{U}$ the set $\text{spm}(x)$ is nonempty, (o)-closed, cyclic and bounded subset of E .*

Proof. First we shall show that $\text{spm}(x)$ is nonempty. Indeed, for $a \in E$ we put

$$\nabla_a = \{\pi \in \nabla : \pi \neq 0, \pi(ae - x) \in \text{Inv}(\pi\mathcal{U})\}.$$

Let $\pi_a = \bigvee \nabla_a$. It is clear that $\pi_a(ae - x) \in \text{Inv}(\pi_a\mathcal{U})$, and for every $\pi \in \nabla, \pi \leq \pi_a^\perp$ one has

$$(2.4) \quad \pi(ae - x) \notin \text{Inv}(\pi\mathcal{U}).$$

Denote $\pi_0 = \bigwedge \{\pi_a : a \in E\}$. Assume that $\pi_0 \neq 0$. Then

$$\pi_0(ae - x) \in \text{Inv}(\pi_0\mathcal{U})$$

for all $a \in E$. But this contradicts to $\text{spm}_{\pi_0\mathcal{U}}(\pi_0x) \neq \emptyset$. Therefore $\pi_0 = 0$. Now we can choose a sequence $(a_n)_{n \in \mathbb{N}} \subset E$ such that $\bigwedge_{n=1}^{\infty} \pi_{a_n} = 0$. Let us define

$$q_1 = \pi_{a_1}^\perp, \quad q_n = \pi_{a_n}^\perp \wedge q_{n-1}^\perp, \quad n > 1$$

and put $a = \sum_{n=1}^{\infty} q_n a_n$. Then $(q_n)_{n \in \mathbb{N}}$ is a partition of unity in ∇ . Take any $\pi \in \nabla, \pi \neq 0$. Then $\pi q_k \neq 0$ for some $k \in \mathbb{N}$. From the definition of q_k one gets $\pi q_k \leq \pi_{a_k}^\perp$. So, from (2.4) it follows that

$$\pi q_k(a_k e - x) \notin \text{Inv}(\pi q_k \mathcal{U}).$$

The equality $\pi q_k a = \pi q_k a_k$ implies that $\pi(ae - x) \notin \text{Inv}(\pi\mathcal{U})$. Hence, $a \in \text{spm}(x)$.

Now let us show $\text{spm}(x)$ is cyclic. Indeed, let $(a_n)_{n \in \mathbb{N}} \subset \text{spm}(x)$, and $(\pi_n)_{n \in \mathbb{N}}$ be a partition of unity in ∇ . Denote $a = \sum_{n=1}^{\infty} \pi_n a_n$. Take any $\pi \in \nabla, \pi \neq 0$. Then $\pi \pi_k \neq 0$ for some $k \in \mathbb{N}$. According to definition of π_k we get

$$\pi \pi_k(a_k e - x) \notin \text{Inv}(\pi \pi_k \mathcal{U}).$$

Since $\pi \pi_k a = \pi \pi_k a_k$, one finds $\pi(ae - x) \notin \text{Inv}(\pi\mathcal{U})$. Hence, $a \in \text{spm}(x)$.

To show the closedness of $\text{spm}(x)$ take a sequence $(a_n)_{n \in \mathbb{N}} \subset \text{spm}(x)$ such that $a_n \xrightarrow{(o)} a$. Assume that $a \notin \text{spm}(x)$. Then there exists $\pi \in \nabla, \pi \neq 0$ such that $\pi(ae - x) \in \text{Inv}(\pi\mathcal{U})$. From $a_n \xrightarrow{(o)} a$, due to Proposition 2.3 (i) one can find $n \in \mathbb{N}$ such that $\pi(a_n e - x) \in \text{Inv}(\pi\mathcal{U})$. This contradicts to $a_n \in \text{spm}(x)$. So, $a \in \text{spm}(x)$.

Finally let us take an arbitrary element $a \in \text{spm}(x)$. Suppose that the set

$$A = \{\omega \in \Omega : |a(\omega)| > \|x\|(\omega)\}$$

has a positive measure. Due to Proposition 2.3 (i) we conclude that $\chi_A(ae - x)$ is invertible in $\chi_A\mathcal{U}$. But this contradicts to $a \in \text{spm}(x)$. Hence $\chi_A = 0$ and $|a| \leq \|x\|$, which implies the boundedness of $\text{spm}(x)$. The proof is complete. \square

3. APPLICATIONS

Next we shall prove a vector version of Gelfand–Mazur’s Theorem.

Theorem 3.1. *Let \mathcal{X} be a measurable bundle of banach algebras over Ω with a lifting. If every element of the algebra $E(\Omega, \mathcal{X})$ with unit support is invertible, then $E(\Omega, \mathcal{X})$ is isometrically isomorphic to E .*

Proof. Let $x \in E(\Omega, \mathcal{X})$. According to Theorem 2.5 there exists $a_x \in \text{spm}(x)$. Let e_x be the support of $a_x e - x$, i.e. e_x is an indicator function of a measurable set $\{\omega \in \Omega : \|a_x e - x\|(\omega) \neq 0\}$. The element $e_x(a_x e - x) + e_x^\perp e$ has unit support, and therefore, it is invertible, i.e. one finds $z \in E(\Omega, \mathcal{X})$ such that

$$(e_x(a_x e - x) + e_x^\perp e)z = e.$$

Whence $e_x(a_x e - x) \in \text{Inv}(e_x E(\Omega, \mathcal{X}))$. By $a_x \in \text{spm}(x)$ one gets $e_x = 0$. This implies that $a_x e - x = 0$, i.e. $a_x e = x$. Due to

$$\|x\| = |a_x| \|e\| = |a_x|,$$

we obtain the mapping $x \mapsto a_x$ is an isometry from $E(\Omega, \mathcal{X})$ onto E . For every $x, y \in E(\Omega, \mathcal{X})$ one has $xy = a_x e a_y e = a_x a_y e$. Hence, the correspondence $x \mapsto a_x$ is isometrically isomorphism from $E(\Omega, \mathcal{X})$ onto E . The proof is complete. \square

Next we are going to prove an other vector version of characterization of the field \mathbb{C} in the setting Banach algebras (see [18, Theorem 10.19]).

Theorem 3.2. *Let \mathcal{X} be a measurable bundle of banach algebras over Ω with a lifting. If there exists $m \in E$ such that $\|x\| \|y\| \leq m \|xy\|$ for all $x, y \in E(\Omega, \mathcal{X})$ then $E(\Omega, \mathcal{X})$ is isometrically isomorphic to E .*

Proof. Let us consider the following two cases.

CASE 1. Let $m \in L^\infty(\Omega)$. Then there exists $c \in \mathbb{R}$ such that $m \leq c \mathbf{1}$. Hence

$$\|x\| \|y\| \leq c \|xy\|$$

for all $x, y \in E(\Omega, \mathcal{X})$.

Let us fix a point $\omega \in \Omega$. Applying the lifting p on $L^\infty(\Omega)$ to the last inequality we obtain

$$p(\|x\|)(\omega) p(\|y\|)(\omega) \leq c p(\|xy\|)(\omega).$$

Taking into account this inequality and property 6 of $\ell_{\mathcal{X}}$ we get

$$\|\ell_{\mathcal{X}}(x)(\omega)\|_{X(\omega)} \|\ell_{\mathcal{X}}(y)(\omega)\|_{X(\omega)} \leq c \|\ell_{\mathcal{X}}(xy)(\omega)\|_{X(\omega)}.$$

This implies that

$$\|x_\omega\|_{X(\omega)} \|y_\omega\|_{X(\omega)} \leq c \|x_\omega y_\omega\|_{X(\omega)}$$

holds for all $x_\omega, y_\omega \in X(\omega)$. According to Theorem 10.19 [18] we conclude that $X(\omega)$ is isomorphic to \mathbb{C} . Now Theorem 2.2 yields that $E(\Omega, \mathcal{X})$ is isomorphic to E . Hence, for each $x \in E(\Omega, \mathcal{X})$ one finds $a_x \in E$ such that $x = a_x e$. The same argument as in the proof of Theorem 3.2 one can show that the correspondence $x \mapsto \lambda_x$ is isometrically isomorphism from $E(\Omega, \mathcal{X})$ onto E .

CASE 2. Let $m \in E$ be arbitrary. Putting $x = y = e$ to the inequality $\|x\| \|y\| \leq m \|xy\|$ implies that $m \geq \mathbf{1}$. For each $n \in \mathbb{N}$ we put

$$\Omega_n = \{\omega \in \Omega : n \leq m(\omega) < n + 1\}, \quad \pi_n = \chi_{\Omega_n}.$$

Then $\bigvee_{n=1}^{\infty} \pi_n = \mathbf{1}$ and $\pi_n m \leq \pi_n(n+1)$ for all $n \in \mathbb{N}$. Hence, for every $x, y \in E(\Omega, \mathcal{X})$, $n \in \mathbb{N}$ one has

$$\pi_n \|x\| \|y\| \leq \pi_n(n+1) \|xy\|.$$

The Case 1 yields that $\pi_n E(\Omega, \mathcal{X})$ is isometrically isomorphic to $\pi_n E$. Due to construction we have $\bigvee_{n=1}^{\infty} \pi_n = \mathbf{1}$ and $\pi_i \pi_j = 0$ ($i \neq j$), which implies that $E(\Omega, \mathcal{X})$ is isometrically isomorphic to E . The proof is complete. \square

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INOMJON GANIEV, DEPARTMENT OF SCIENCE IN ENGINEERING, FACULTY OF ENGINEERING, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. Box 10, 50728, KUALA-LUMPUR, MALAYSIA

E-mail address: inam@iium.edu.my, ganiev1@rambler.ru

FARRUKH MUKHAMEDOV, DEPARTMENT OF COMPUTATIONAL & THEORETICAL SCIENCES, FACULTY OF SCIENCE, INTERNATIONAL ISLAMIC UNIVERSITY MALAYSIA, P.O. Box, 141, 25710, KUANTAN, PAHANG, MALAYSIA

E-mail address: far75m@yandex.ru, farrukh_m@iium.edu.my

(Karimbergen Kudaybergenov) DEPARTMENT OF MATHEMATICS, KARAKALPAK STATE UNIVERSITY, 230113 NUKUS, UZBEKISTAN.

E-mail address: karim2006@mail.ru